GRANIČNE VREDNOSTI FUNKCIJA
zadaci II deo

U sledećim zadacima ćemo koristiti poznatu graničnu vrednost:

$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}$ ali i manje "varijacije''

$\lim_{x\to 0}\frac{\sin ax}{ax}=1$ i $\lim_{x\to 0}\frac{\sin^n ax}{(ax)^n}=1$

Zadaci:

1) Odrediti sledeće granične vrednosti:

a) $\lim_{x\to 0}\frac{\sin 4x}{x}$;

b) $\lim_{x\to 0}\frac{tgx}{x}$;

v) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$;

g) $\lim_{x\to a}\frac{\sin x-\sin a}{x-a}$;

Rešenja:

a) $\lim_{x\to 0}\frac{\sin 4x}{x}$;(i gore i dole dodamo 4) $=\lim_{x\to 0}\frac{4\sin 4x}{4x}=4\cdot\boxed{\lim_{x\to 0}\frac{\sin 4x}{4x}}=4\cdot 1=4$

Ovde smo " napravili" i upotrebili da je $\lim_{x\to 0}\frac{\sin ax}{ax}=1$

b) $\lim_{x\to 0}\frac{tgx}{x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=\lim_{x\to 0}\boxed{\frac{\sin x}{x}}\cdot\frac{1}{\cos x}=\lim_{x\to 0}1\cdot\frac{1}{\cos x}$

$=1\cdot\lim_{x\to 0}\frac{1}{\cos x}=1\cdot\frac{1}{\cos 0}=1\cdot\frac{1}{1}=1$

v) $\lim_{x\to 0}\frac{1-\cos x}{x^2}=$ iskoristićemo formulu iz trigonometrije: $1-\cos x=2\sin^2\frac{x}{2}$

$$=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=(\text{ dodamo }4)=\lim_{x\to 0}2\cdot\frac{\sin^2\frac{x}{2}}{4\frac{x^2}{4}}=\frac{2}{4}\cdot\boxed{\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}}=\frac{1}{2}\cdot 1=\frac{1}{2}$$

g) $\lim\limits_{x\to a} = \dfrac{\sin x - \sin a}{x-a} =$ iskoristićemo formulu (pogledaj PDF fajl iz II godine)

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$= \lim_{x\to a}\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a} = \text{malo prisredimo...}$$

$$= \lim_{x\to a}\cos\frac{x+a}{2}\cdot\boxed{\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}} = \lim_{x\to a}\cos\frac{x+a}{2}\cdot 1 =$$

$$= \cos\frac{a+a}{2} = \cos\frac{2a}{2} = \cos a$$

2) Izračunati sledeće granične vrednosti:

a) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{\sqrt{x+1}-1}$;

b) $\lim\limits_{x\to \pi}\dfrac{\cos\frac{x}{2}}{x-\pi}$;

v) $\lim\limits_{x\to 1}\dfrac{\sin(1-x)}{\sqrt{x}-1}$;

a)

$$\lim_{x\to 0}\frac{\sin 4x}{\sqrt{x+1}-1} = \text{najpre racionalizacija}$$

$$= \lim_{x\to 0}\frac{\sin 4x}{\sqrt{x+1}-1}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} =$$

$$= \lim_{x\to 0}\frac{\sin 4x\cdot\left(\sqrt{x+1}+1\right)}{x+1-1} = \lim_{x\to 0}\frac{\sin 4x\cdot\left(\sqrt{x+1}+1\right)}{x}$$

sad i gore i dole dodamo 4

$$= \lim_{x\to 0}\frac{4\sin 4x\left(\sqrt{x+1}+1\right)}{4x} = \lim_{x\to 0}4\boxed{\frac{\sin 4x}{4x}}\left(\sqrt{x+1}+1\right) = \lim_{x\to 0}4\cdot 1\cdot\left(\sqrt{x+1}+1\right) =$$

$$= 4\left(\sqrt{0+1}+1\right) = 4\cdot 2 = 8$$

b)

$$\lim_{x\to\pi}\frac{\cos\frac{x}{2}}{x-\pi}=\text{ovde ćemo najpre uzeti smenu: } x-\pi=t,\text{, pa kad } x\to\pi\text{, onda } t\to\pi-\pi=0,\text{ dakle } \boxed{t\to 0}$$

$$\lim_{t\to0}\frac{\cos\frac{t+\pi}{2}}{t}=\lim_{t\to0}\frac{\cos\left(\frac{\pi}{2}+\frac{t}{2}\right)}{t}=\lim_{t\to0}\frac{-\sin\frac{t}{2}}{t}\quad \left(\text{ jer je }\cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha\ \right)$$

$$=-\lim_{t\to0}\frac{\sin\frac{t}{2}}{2\cdot\frac{t}{2}}=-\lim_{t\to0}\frac{1}{2}\boxed{\frac{\sin\frac{t}{2}}{\frac{t}{2}}}=-\frac{1}{2}\cdot1=-\frac{1}{2}$$

v)

$$\lim_{x\to1}\frac{\sin(1-x)}{\sqrt{x}-1}=\text{ najpre racionalizacija}$$

$$\lim_{x\to1}\frac{\sin(1-x)}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\lim_{x\to1}\frac{\sin(1-x)(\sqrt{x}+1)}{x-1}=\text{ sada smena } x-1=t,\text{ kad } x\to1\text{ tad } t\to0$$

$$=\lim_{t\to0}\frac{\sin(-t)\left(\sqrt{t+1}+1\right)}{t}=\lim_{t\to0}\frac{-\sin(t)\left(\sqrt{t+1}+1\right)}{t}=-\lim_{t\to0}\boxed{\frac{\sin t}{t}}\left(\sqrt{t+1}+1\right)$$

$$=-\lim_{t\to0}1\cdot\left(\sqrt{t+1}+1\right)=-(1+1)=-2$$

U sledećim zadacima ćemo koristiti:

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e}\qquad \text{I}\qquad \lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^{ax}=e$$

Još nam treba i činjenica da je e^x neprekidna funkcija i važi:

$$\boxed{\lim_{x\to a}e^{f(x)}=e^{\lim\limits_{x\to a}f(x)}}$$

3) Odrediti sledeće granične vrednosti:

a) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x$;

b) $\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x$;

c) $\lim_{x\to\infty}x(\ln(x+1)-\ln x)$;

Rešenja:

a)

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \text{ovde gde je 3 mora biti 1, pa ćemo 3 ‘spustiti’ ispod x}$$

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^x = \text{ sad kod } x \text{ u eksponentu pomnožimo i podelimo sa3}$$

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^x = = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}\cdot 3} = \boxed{\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}}}^{3} = e^3$$

b)

$$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x = \textbf{trik}\text{: u zagradi ćemo dodati 1 i oduzeti 1} =$$

$$\lim_{x\to\infty}\left(1+\frac{x+1}{x-1}-1\right)^x = \lim_{x\to\infty}\left(1+\frac{x+1-1(x-1)}{x-1}\right)^x = \lim_{x\to\infty}\left(1+\frac{x+1-x+1}{x-1}\right)^x$$

$$= \lim_{x\to\infty}\left(1+\frac{2}{x-1}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}\cdot\frac{2}{x-1}\cdot x} =$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}\cdot\frac{2x}{x-1}} = \boxed{\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}}}^{\frac{2x}{x-1}} = \lim_{x\to\infty} e^{\frac{2x}{x-1}} = e^{\lim_{x\to\infty}\frac{2x}{x-1}} = e^2$$

v)

$$\lim_{x\to\infty} x\cdot(\ln(x+1)-\ln x) = \lim_{x\to\infty}[x\cdot\ln\frac{x+1}{x}] = \lim_{x\to\infty}\ln\left(\frac{x+1}{x}\right)^x =$$

(pošto je ln neprekidna funkcija i ona može da zameni mesto sa lim)

$$\ln\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = \ln\lim_{x\to\infty}\left(\frac{x}{x}+\frac{1}{x}\right)^x = \ln\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \ln e = 1$$

Ovde smo koristili pravila(pogledaj II godina logaritmi): lnA - lnB = $\ln\frac{A}{B}$ i $n\cdot\ln A = \ln A^n$

4) Odrediti sledeće granične vrednosti:

a) $\lim_{x\to 0}(1+3tg^2x)^{ctg^2x}=?$

b) $\lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}}=?$

Rešenja:

a) $\lim_{x\to 0}(1+3tg^2x)^{ctg^2x}=?$

$$\lim_{x\to 0}(1+3tg^2x)^{ctg^2x}=\lim_{x\to 0}(1+3\cdot\frac{1}{ctg^2x})^{ctg^2x}=\lim_{x\to 0}(1+\frac{1}{\frac{ctg^2x}{3}})^{ctg^2x}=$$

$$\lim_{x\to 0}(1+\frac{1}{\frac{ctg^2x}{3}})^{ctg^2x\cdot\frac{3}{3}}=\lim_{x\to 0}(1+\frac{1}{\frac{ctg^2x}{3}})^{\frac{ctg^2x}{3}\cdot 3}=\boxed{\lim_{x\to 0}(1+\frac{1}{\frac{ctg^2x}{3}})^{\frac{ctg^2x}{3}}}^{3}=e^3$$

b) $\lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}}=?$

Najpre ćemo dodati i oduzeti jedinicu…

$$\lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}}=\lim_{x\to 0}(1+\cos x-1)^{\frac{1}{\sin^2 x}}$$

Dalje moramo upotrebiti formulicu: $1-\cos x=2\sin^2\frac{x}{2}$

$$\lim_{x\to0}(\cos x)^{\frac{1}{\sin^2 x}}=\lim_{x\to0}(1+\cos x-1)^{\frac{1}{\sin^2 x}}=\lim_{x\to0}(1-(1-\cos x))^{\frac{1}{\sin^2 x}}=\lim_{x\to0}(1-2\sin^2\frac{x}{2})^{\frac{1}{\sin^2 x}}=\lim_{x\to0}\left(1\boxed{-}\frac{1}{\frac{1}{2\sin^2\frac{x}{2}}}\right)^{\frac{1}{\sin^2 x}}=$$

$$=\lim_{x\to0}\left(1+\frac{1}{\boxed{-}\frac{1}{2\sin^2\frac{x}{2}}}\right)^{\frac{1}{\sin^2 x}}=\{formula\ \ \sin^2 x=4\sin^2\frac{x}{2}\cos^2\frac{x}{2}\}=\lim_{x\to0}\left(1+\frac{1}{-\frac{1}{2\sin^2\frac{x}{2}}}\right)^{\frac{1}{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}}=\lim_{x\to0}\left(1+\frac{1}{-\frac{1}{2\sin^2\frac{x}{2}}}\right)^{\frac{1}{\sin^2\frac{x}{2}}\cdot\frac{-1}{4\cos^2\frac{x}{2}}}=$$

$$=\boxed{\lim_{x\to0}\left(1+\frac{1}{-\frac{1}{2\sin^2\frac{x}{2}}}\right)^{\frac{1}{2\sin^2\frac{x}{2}}}}^{\frac{-1}{2\cos^2\frac{x}{2}}}=e^{\lim_{x\to0}\frac{-1}{2\cos^2\frac{x}{2}}}=e^{-\frac{1}{2}}$$